

# Wind Estimation using an $H_\infty$ Filter with Fixed-Wing Aircraft Flight Test Results

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**Indirect wind estimation onboard unmanned aerial systems (UASs) can be accomplished using existing air vehicle sensors along with a dynamic model of the UAS augmented with additional wind-related states. It is often desired to extract a mean component of the wind the from frequency fluctuations (i.e. turbulence). Commonly, a variation of the KALMAN filter is used, with explicit or implicit assumptions about the nature of the random wind velocity. This paper presents an H-infinity ( $H_\infty$ ) filtering approach to wind estimation which requires no assumptions about the statistics of the process or measurement noise. To specify the wind frequency content of interest a low-pass filter is incorporated. We develop the augmented UAS model in continuous-time, derive the  $H_\infty$  filter, and introduce a KALMAN filter for comparison. The filters are applied to data gathered during UAS flight tests and validated using a vaned air data unit onboard the aircraft. The  $H_\infty$  filter provides quantitatively better estimates of the wind than the KALMAN filter, with approximately 50% less root-mean-square (RMS) error in the majority of cases.**

## I. Introduction

WIND estimation from air vehicle motion is useful for many reasons, including weather forecasting and air traffic management [1]. The ability to estimate wind indirectly (i.e. without specialized instrumentation) is of particular interest due to the increasing ubiquity of small unmanned aerial systems (UASs) [2] and the unobtrusive/low-cost nature of using operational sensors (e.g. air vehicle GPS, accelerometers, and gyroscopes). A common goal is to extract a mean component of the wind from higher frequency fluctuations (gusts or atmospheric turbulence). For example the United States' Automated Surface Observation System (ASOS) [3], which provides crucial meteorological data to the National Weather Service (NWS), the Federal Aviation Administration (FAA), and the Department of Defense (DOD), reports 2-minute averaged winds which are calculated from sets of 5-second averaged winds [4].

In many UAS-based wind estimation applications, as surveyed in [5], some variation of the well-known KALMAN filter is used as a state estimator to determine the mean wind. Examples are a two-stage KALMAN filter in [6], an unscented KALMAN filter in [7], and the traditional linear KALMAN filter in [8]. The KALMAN filter is the minimum covariance linear filter for a perfectly known system model subject to white, random processes (measurement and/or process noise) with known mean and covariance [9]. In the case where the goal is to estimate local bulk fluid flow in a wind field, the higher frequency fluctuations cannot be well-approximated as a GAUSSIAN, white random process. Traditionally, this has been overcome by using a shaping filter such as the DRYDEN turbulence model which is driven by GAUSSIAN, white noise. However, such approaches require *a priori* knowledge of certain atmospheric characteristics. Often, this information is either unknown or the turbulence models do not sufficiently capture the atmospheric dynamics. This is especially true at low altitudes and in urban environments where wind data are especially valuable for establishing boundary conditions used in numerical weather predictions [10]. To solve this problem, this paper presents the  $H_\infty$  filtering approach to wind estimation, which makes no assumptions about the statistics of the disturbance, but instead assumes the system is subject to arbitrary finite-energy exogenous inputs. The optimal  $H_\infty$  filter does not aim to minimize estimate error in a quadratic sense, but rather minimizes the worst-case map from disturbances to estimate error. Limiting the worst-case error is a useful feature of a wind estimator for the applications mentioned. The  $H_\infty$  approach is thus an attractive alternative to the KALMAN filter for the problem of estimating bulk wind flow in turbulent conditions.

In Section II, we present the aircraft motion model for which we evaluate our estimation filters. In Section III we develop the  $H_\infty$  filter. Section IV presents the steps involved in the synthesis of the  $H_\infty$  filter and introduces the KALMAN

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filter used for comparison. In Section V we present the results of applying the two filters to experimental flight test data. Lastly, in Section VI we present our conclusions and future work.

## II. Aircraft Motion in Wind

### A. Wind Field

Consider an aircraft in a wind velocity field,  $\mathbf{W}$ , that varies in space and time. At an instant of time, the wind velocity that would exist at the aircraft center of gravity (CG) if the aircraft were absent, the apparent wind, is

$$\mathbf{w}(t) = \mathbf{W}(\mathbf{X}(t), t) \quad (1)$$

where  $\mathbf{X} = [X_N \ X_E \ X_D]^\top$  is the north-east-down position of the aircraft CG in an orthonormal Earth-fixed inertial reference frame  $\mathcal{F}_I$ , defined by unit vectors  $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ . The total time derivative of the apparent wind is

$$\frac{d\mathbf{w}}{dt} = \frac{\partial \mathbf{W}(\mathbf{X}, t)}{\partial t} + \frac{\partial \mathbf{W}(\mathbf{X}, t)}{\partial \mathbf{X}} \frac{d\mathbf{X}}{dt} \quad (2)$$

Adopting the standard overdot notation,  $(\dot{\cdot})$ , for the total time derivative we have

$$\dot{\mathbf{w}} = \frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{W}}{\partial \mathbf{X}} \dot{\mathbf{X}} \quad (3)$$

Depending on the context, we adopt one of the following two assumptions about the dynamics of the wind field.

#### Assumption 1

- (a) *The wind field evolves slowly compared to the aircraft velocity and is thus considered frozen, consistent with TAYLOR's hypothesis [11], such that  $\frac{\partial \mathbf{W}}{\partial t} = \mathbf{0}$ .*
- (b) *The variation of the wind field with position is negligible such that  $\frac{\partial \mathbf{W}}{\partial \mathbf{X}} = \mathbf{0}$ .*

The above assumption leads to the conclusion that  $\dot{\mathbf{w}} = \mathbf{0}$  and the air relative angular velocity is equal to the vehicle angular velocity. This assumption, which is consistent with [1, 12], could be relaxed but it simplifies the results presented here for the  $H_\infty$  filter.

Assumption 1 may seem incompatible with the motivating challenge of estimating variations in the wind velocity. However, the assumption of a uniform wind field does not impede the model-based filter from adjusting wind velocity estimates based on measurements.

**Assumption 2** *The wind is a superposition of a bulk flow velocity (constant or slowly varying),  $\bar{\mathbf{w}}$ , and a higher frequency, finite-energy disturbance (gusts or turbulence),  $\delta\mathbf{w}$ , so that:*

$$\mathbf{w}(t) = \bar{\mathbf{w}}(t) + \delta\mathbf{w}(t) \quad (4)$$

Eq. (4) is known as a REYNOLDS decomposition [13] of the wind.

### B. Aircraft Nonlinear Equations of Motion in Wind

Having introduced the dynamics of the wind field in Section II.A, we now briefly present the aircraft equations of motion relative to the wind. The reader is directed to [14] and [15] for a full treatment of the derivation. We begin by defining the orthonormal aircraft body reference frame,  $\mathcal{F}_B$ , defined by unit vectors  $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$  originating at the CG, where  $\mathbf{b}_1$  points forward,  $\mathbf{b}_2$  points out the right wing, and  $\mathbf{b}_3$  completes the right-hand rule. The transformation from  $\mathcal{F}_B$  to  $\mathcal{F}_I$  is given by the rotation matrix

$$\mathbf{R}_{IB}(\boldsymbol{\Theta}) = e^{[\mathbf{e}_3 \times] \psi} e^{[\mathbf{e}_2 \times] \theta} e^{[\mathbf{e}_1 \times] \phi} \quad (5)$$

where  $\boldsymbol{\Theta} = [\phi \ \theta \ \psi]^\top$  are the aircraft Euler angles. Here,  $\mathbf{e}_1 = [1 \ 0 \ 0]^\top$ , etc., and  $[(\cdot) \times]$  is the skew-symmetric cross product equivalent matrix satisfying  $[\mathbf{a} \times] \mathbf{b} = \mathbf{a} \times \mathbf{b}$ . Note that  $\mathbf{R}_{BI}(\boldsymbol{\Theta}) := \mathbf{R}_{IB}^{-1}(\boldsymbol{\Theta}) = \mathbf{R}_{IB}^\top(\boldsymbol{\Theta})$ . The aircraft kinematics are given by

$$\dot{\mathbf{X}} = \mathbf{R}_{IB}(\boldsymbol{\Theta}) \mathbf{v} \quad (6)$$

$$\dot{\mathbf{R}}_{IB}(\boldsymbol{\Theta}) = \mathbf{R}_{IB}(\boldsymbol{\Theta})[\boldsymbol{\omega} \times] \quad (7)$$

where  $\mathbf{v} = [u \ v \ w]^\top$  is the velocity with respect to the inertial frame, given in the body frame, and  $\boldsymbol{\omega} = [p \ q \ r]^\top$  is the angular velocity with respect to the inertial frame, expressed in the body frame. Lastly, as an alternative to Eq. (7), the Euler angle rate equations are

$$\dot{\boldsymbol{\Theta}} = \mathbf{L}_{IB}(\boldsymbol{\Theta})\boldsymbol{\omega} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad (8)$$

As in [16] we make the assertion that the aerodynamic forces,  $\mathbf{F}$ , and moments,  $\mathbf{M}$ , depend of the velocity of the aircraft relative to the wind. Let

$$\mathbf{v}_r = \mathbf{v} - \mathbf{R}_{BI}(\boldsymbol{\Theta})\mathbf{w} \quad (9)$$

where  $\mathbf{v}_r = [u_r \ v_r \ w_r]^\top$  is the velocity of the body's center of mass with respect to the wind, given in the body frame. Eq. (9) is known as the wind triangle. From Assumption 1, the air-relative angular velocity of the aircraft is equal to the total angular velocity,  $\boldsymbol{\omega}$ . Therefore the aerodynamic forces and moments have the functional form

$$\mathbf{F} = \mathbf{F}(\mathbf{v}_r, \boldsymbol{\omega}, \mathbf{u}), \quad \mathbf{M} = \mathbf{M}(\mathbf{v}_r, \boldsymbol{\omega}, \mathbf{u}) \quad (10)$$

where  $\mathbf{u} = [T_{\eta_0} \ \delta_a \ \delta_e \ \delta_r]^\top$  are the throttle, aileron, elevator, and rudder control inputs, respectively. We use an aerodynamic model from [17] with non-dimensional coefficients of the form

$$C_X = C_{X_o} + C_{X_\alpha} \alpha + C_{X_{\alpha^2}} \alpha^2 + C_{X_{\delta_e}} \delta_e + C_T T_{\eta_0} \quad (11a)$$

$$C_Z = C_{Z_o} + C_{Z_\alpha} \alpha + C_{Z_q} \hat{q} \quad (11b)$$

$$C_m = C_{m_o} + C_{m_\alpha} \alpha + C_{m_q} \hat{q} + C_{m_{\delta_e}} \delta_e \quad (11c)$$

$$C_Y = C_{Y_\beta} \beta + C_{Y_p} \hat{p} + C_{Y_r} \hat{r} + C_{Y_{\delta_a}} \delta_a + C_{Y_{\delta_r}} \delta_r \quad (11d)$$

$$C_l = C_{l_\beta} \beta + C_{l_p} \hat{p} + C_{l_{\delta_a}} \delta_a \quad (11e)$$

$$C_n = C_{n_\beta} \beta + C_{n_r} \hat{r} + C_{n_{\delta_a}} \delta_a + C_{n_{\delta_r}} \delta_r \quad (11f)$$

where the propulsion coefficient,  $C_T$ , is modeled using information from [18]. Note that the overhats ( $\hat{\cdot}$ ) indicate nondimensional angular rates. Here the angle-of-attack,  $\alpha$ , and angle-of-sideslip,  $\beta$ , define the transformation between the body and wind-relative frames with

$$\alpha = \tan^{-1} \left( \frac{w_r}{u_r} \right) \quad \text{and} \quad \beta = \sin^{-1} \left( \frac{v_r}{V_r} \right) \quad (12)$$

where  $V_r = \|\mathbf{v}_r\|$ . Thus, the well established nonlinear equations of motion, covered at length in [14, 19], adapted for the assumed wind field are

$$\dot{\mathbf{X}} = \mathbf{R}_{IB}(\boldsymbol{\Theta})\mathbf{v}_r + \mathbf{w} \quad (13a)$$

$$\dot{\boldsymbol{\Theta}} = \mathbf{L}_{IB}(\boldsymbol{\Theta})\boldsymbol{\omega} \quad (13b)$$

$$\dot{\mathbf{v}}_r = \mathbf{v}_r \times \boldsymbol{\omega} + g\mathbf{R}_{BI}(\boldsymbol{\Theta})\mathbf{e}_3 + \frac{1}{m}\mathbf{F}(\mathbf{v}_r, \boldsymbol{\omega}, \mathbf{u}) \quad (13c)$$

$$\dot{\boldsymbol{\omega}} = \mathbf{I}^{-1}(\mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \mathbf{M}(\mathbf{v}_r, \boldsymbol{\omega}, \mathbf{u})) \quad (13d)$$

where  $\mathbf{I}$  is the aircraft moment of inertia tensor defined at the CG in  $\mathcal{F}_B$ . At last, we define the aircraft state and measurement equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \quad (14a)$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{v}) \quad (14b)$$

where the state, input, and wind vectors are

$$\mathbf{x} = \left[ \mathbf{X}^\top \ \boldsymbol{\Theta}^\top \ \mathbf{v}_r^\top \ \boldsymbol{\omega}^\top \right]^\top, \quad \mathbf{u} = \left[ T_{\eta_0} \ \delta_a \ \delta_e \ \delta_r \right]^\top, \quad \mathbf{w} = \left[ w_N \ w_E \ w_D \right]^\top \quad (15)$$

and where  $\mathbf{v}$  is an unknown finite-energy measurement disturbance (i.e. noise), which is described further in Section III. We will now present the linearized equations of motion for the aircraft in a wind field.

### C. Aircraft Linear Equations of Motion

It is assumed that, for the mission of estimating the wind, the aircraft maintains an equilibrium flight condition over a desired time or distance. Therefore, we can define an equilibrium and any perturbations from that equilibrium not attributed to control inputs or initial conditions can be ascribed to the wind. To prepare for the formulation of the  $H_\infty$  filter given in Section III, it is necessary to develop a linear time-invariant (LTI) model from the equations of motion given in Section II.B. Define

$$A = \left. \frac{\partial f}{\partial \mathbf{x}} \right|_{\mathbf{x}_e, \mathbf{u}_e, \mathbf{w}_e} \quad B = \left. \frac{\partial f}{\partial \mathbf{u}} \right|_{\mathbf{x}_e, \mathbf{u}_e, \mathbf{w}_e} \quad \Pi = \left. \frac{\partial f}{\partial \mathbf{w}} \right|_{\mathbf{x}_e, \mathbf{u}_e, \mathbf{w}_e} \quad (16)$$

where  $\mathbf{x}_e$ ,  $\mathbf{u}_e$ , and  $\mathbf{w}_e$  are the equilibrium states, controls, and wind vectors that define a steady flight condition (e.g. straight and level cruise in still air). The linear perturbation equations are

$$\Delta \dot{\mathbf{x}} = A\Delta \mathbf{x} + B\Delta \mathbf{u} + \Pi\Delta \mathbf{w}, \quad \Delta \mathbf{x}(0) = \Delta \mathbf{x}_0 \quad (17a)$$

$$\mathbf{y} = C\Delta \mathbf{x} + D\Delta \mathbf{u} + \Lambda \mathbf{v} \quad (17b)$$

where  $\Delta \mathbf{x}$ ,  $\Delta \mathbf{u}$ , and  $\Delta \mathbf{w}$  are perturbations from the equilibrium state, control, and wind vectors, and  $\mathbf{v}$  is measurement noise. For brevity, we will drop the  $\Delta$  notation. With the linear motion model defined, we may now develop the continuous-time  $H_\infty$  filter.

## III. Steady-State $H_\infty$ Filter

In this section we first present a low-pass filter, which is used to define  $\bar{\mathbf{w}}$  – the slowly-varying component of the faster fluctuating wind which we want to estimate. Next, we describe the inclusion of the frequency filter within the  $H_\infty$  filter framework as described in [20]. Following the development of the  $H_\infty$  filter, we provide some guidance for choosing the design parameters of the overall filter. Then, we introduce a KALMAN filter against which we compare the  $H_\infty$  filter.

### A. Low-pass Filter

As was motivated in Section I, we often wish to estimate the mean wind in the presence of disturbances. Thus, recalling from Assumption 2 that  $\mathbf{w} = \bar{\mathbf{w}} + \delta \mathbf{w}$ , we can design a low-pass filter to isolate the wind fluctuations of interest, given by the transfer function  $\mathbf{H}_{\bar{\mathbf{w}}}(s)$  such that

$$\bar{\mathbf{w}}(s) := \mathbf{H}_{\bar{\mathbf{w}}}(s)\mathbf{w}(s) \quad (18)$$

Suppose Eq. (18) has the *state-space* realization

$$\dot{\boldsymbol{\zeta}} = \mathbf{A}_{\bar{\mathbf{w}}} \boldsymbol{\zeta} + \mathbf{B}_{\bar{\mathbf{w}}} \mathbf{w} \quad (19a)$$

$$= \mathbf{A}_{\bar{\mathbf{w}}} \boldsymbol{\zeta} + \mathbf{B}_{\bar{\mathbf{w}}} (\bar{\mathbf{w}} + \delta \mathbf{w}) \quad (19b)$$

$$\bar{\mathbf{w}} = \mathbf{C}_{\bar{\mathbf{w}}} \boldsymbol{\zeta} \quad (19c)$$

where  $\boldsymbol{\zeta}$  is an intermediate state in the map from the wind input to the output of interest,  $\bar{\mathbf{w}}$ . We can now augment the system in Eq. (17) to achieve a system of the form

$$\dot{\mathbf{x}}_A = \underbrace{\begin{bmatrix} \mathbf{A} & \Pi \mathbf{C}_{\bar{\mathbf{w}}} \\ \mathbf{0} & \mathbf{A}_{\bar{\mathbf{w}}} + \mathbf{B}_{\bar{\mathbf{w}}} \mathbf{C}_{\bar{\mathbf{w}}} \end{bmatrix}}_{\mathbf{A}_A} \mathbf{x}_A + \underbrace{\begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}}_{\mathbf{B}_A} \mathbf{u} + \underbrace{\begin{bmatrix} \Pi \\ \mathbf{B}_{\bar{\mathbf{w}}} \end{bmatrix}}_{\Pi_A} \delta \mathbf{w}, \quad \mathbf{x}_A(0) = \mathbf{x}_{A_0} \quad (20a)$$

$$\mathbf{y}_A = \underbrace{\begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix}}_{\mathbf{C}_A} \mathbf{x}_A + D\mathbf{u} + \Lambda \mathbf{v} \quad (20b)$$

where the augmented state is  $\mathbf{x}_A = [\mathbf{x}^\top \ \boldsymbol{\zeta}^\top]^\top$ . With the low-pass filter-augmented system defined, we can now discuss the  $H_\infty$  filter.



## B. $H_\infty$ Filter Theory

The exogenous input to the augmented dynamics in Eq. (20) is  $\eta := [\delta \mathbf{w}^\top \ \nu^\top]^\top$ , so that

$$\dot{\mathbf{x}}_A = \mathbf{A}_A \mathbf{x}_A + \mathbf{B}_A \mathbf{u} + \underbrace{\begin{bmatrix} \mathbf{\Pi} & \mathbf{0} \\ \mathbf{B}_{\bar{\mathbf{w}}} & \mathbf{0} \end{bmatrix}}_{\mathbf{G}} \eta \quad (21a)$$

$$\mathbf{y}_A = \mathbf{C}_A \mathbf{x}_A + \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{\Lambda} \end{bmatrix}}_{\mathbf{F}} \eta \quad (21b)$$

$$\mathbf{z} = \underbrace{\begin{bmatrix} \mathbf{L}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\bar{\mathbf{w}}} \end{bmatrix}}_{\mathbf{L}} \mathbf{x}_A \quad (21c)$$

where  $\mathbf{z}$  are the outputs of interest when designing a filter that minimizes the worst-case effect of the exogenous disturbances, as will be shown presently. A LUENBERGER-like filter for the system Eq. (21) is of the form

$$\dot{\hat{\mathbf{x}}}_A = \mathbf{A}_A \hat{\mathbf{x}}_A + \mathbf{B}_A \mathbf{u} + \mathbf{K} (\mathbf{y}_A - \mathbf{C}_A \hat{\mathbf{x}}_A), \quad \hat{\mathbf{x}}_A(0) = \hat{\mathbf{x}}_{A_0} \quad (22a)$$

$$\hat{\mathbf{z}} = \mathbf{L} \hat{\mathbf{x}}_A \quad (22b)$$

where the hat notation,  $(\hat{\cdot})$ , denotes an estimate, and  $\mathbf{K}$  is a generic filter gain matrix to be determined. Now define  $\tilde{\mathbf{x}}_A = \mathbf{x}_A - \hat{\mathbf{x}}_A$ . Thus the error dynamics of the filter are

$$\dot{\tilde{\mathbf{x}}}_A = (\mathbf{A}_A - \mathbf{K} \mathbf{C}_A) \tilde{\mathbf{x}}_A + (\mathbf{G} - \mathbf{K} \mathbf{F}) \eta \quad (23a)$$

$$\tilde{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{L} \tilde{\mathbf{x}}_A \quad (23b)$$

Let  $\mathbf{H}_{\tilde{\mathbf{z}}\eta}(s)$  represent the transfer function from exogenous inputs,  $\eta$ , to estimate error,  $\tilde{\mathbf{z}}$  for the system (23). Also, recall from linear systems theory [21] that the  $H_\infty$  norm of system  $\mathbf{H}_{\tilde{\mathbf{z}}\eta}(s)$  is given by

$$\|\mathbf{H}_{\tilde{\mathbf{z}}\eta}(s)\|_\infty = \text{ess sup}_{\omega_u \in \mathbb{R}} \sigma_{\max}(\mathbf{H}_{\tilde{\mathbf{z}}\eta}(j\omega_u)) \quad (24)$$

where  $\sigma_{\max}$  denotes the largest singular value. Additionally, define  $\mathcal{L}_2[0, \infty)$  as the space of square-integrable functions with bounded energy such that

$$\eta \in \mathcal{L}_2[0, \infty) \quad \Rightarrow \quad \int_0^\infty \|\eta(t)\|_2^2 dt < \infty \quad (25)$$

From [20], we apply the following lemma:

**Lemma 1** (Adapted from Lemma 7.1(c) in [20]) For all  $\eta \in \mathcal{L}_2[0, \infty)$ , for a given  $\gamma > 0$  there exists a linear causal filter such that  $\mathbf{H}_{\tilde{\mathbf{z}}\eta}(s)$  is asymptotically stable and  $\|\mathbf{H}_{\tilde{\mathbf{z}}\eta}(s)\|_\infty < \gamma$  if and only if the algebraic RICCATI inequality (ARI)

$$\mathbf{A}_A^\top \mathbf{X} + \mathbf{X} \mathbf{A}_A - \mathbf{Y} \mathbf{C}_A - \mathbf{C}_A^\top \mathbf{Y}^\top + \gamma^{-2} \mathbf{L}^\top \mathbf{L} + (\mathbf{X} \mathbf{G} - \mathbf{Y} \mathbf{F})(\mathbf{X} \mathbf{G} - \mathbf{Y} \mathbf{F})^\top < \mathbf{0} \quad (26)$$

has a solution  $\mathbf{X} = \mathbf{X}^\top > \mathbf{0}$ , with  $\mathbf{Y} \in \mathbb{R}^{n \times m}$  where  $n$  is the number of states and  $m$  is the number of measurements. Applying the SCHUR complement to Eq. (26) yields the linear matrix inequality

$$\mathcal{Y}(\mathbf{X}, \mathbf{Y}) = \begin{bmatrix} \mathbf{A}_A^\top \mathbf{X} + \mathbf{X} \mathbf{A}_A - \mathbf{Y} \mathbf{C}_A - \mathbf{C}_A^\top \mathbf{Y}^\top + \gamma^{-2} \mathbf{L}^\top \mathbf{L} & \mathbf{X} \mathbf{G} - \mathbf{Y} \mathbf{F} \\ \mathbf{G}^\top \mathbf{X}^\top - \mathbf{F}^\top \mathbf{Y}^\top & -\mathbf{I} \end{bmatrix} < \mathbf{0} \quad (27a)$$

$$\mathbf{X} > \mathbf{0} \quad (27b)$$

Finally, the  $H_\infty$  filter gain,  $\mathbf{K}_\infty$ , is

$$\mathbf{K}_\infty = \mathbf{X}^{-1} \mathbf{Y} \quad (28)$$

Lemma 1 is valid for arbitrary finite-energy disturbances. As described in [20], this is in contrast to traditional minimum variance filters, e.g., the well-known KALMAN filter. The  $H_\infty$  filter accepts noise as a deterministic disturbance with no *a priori* statistical knowledge required (e.g. mean, covariance, normality, whiteness, etc.), save for the assumption of bounded energy. This makes the  $H_\infty$  filter well-suited for applications where little knowledge about the noise statistics is available, such as in the present problem of wind estimation.

## IV. Filter Synthesis for Wind Estimation

### A. Research Aircraft

The research aircraft used in this effort is the My Twin Dream (MTD), shown in Fig. 1, manufactured by My Fly Dream. It is a radio-controlled foam aircraft with counter-rotating twin electric motors and APC 10-in. diameter, 6-in. pitch (10x6) propellers. The aircraft was instrumented with a CubePilot CubeOrange flight computer running PX4 firmware. The sensors onboard the aircraft include triple-redundant accelerometers, a gyroscope, two magnetometers, a Real-Time Kinematic (RTK) global positioning system (GPS) receiver, and a vaned air data unit for validation. The MTD was chosen for its simple construction, propeller location (to accommodate the air data boom), and endurance of approximately 25 minutes. The MTD's physical properties from [17] are listed in Table 1. The aerodynamic force and moment coefficients in Eq. (11) are given in [17]. The propulsion coefficient comes from a model based on the wind tunnel data from [18].

**Table 1** My Twin Dream (MTD) properties [17]

| Property                | Symbol           | Value       | Units             |
|-------------------------|------------------|-------------|-------------------|
| Mass                    | $m$              | 3.311       | kg                |
| Mean aerodynamic chord  | $\bar{c}$        | 0.254       | m                 |
| Projected wing span     | $b$              | 1.800       | m                 |
| Wing reference area     | $S$              | 0.457       | m <sup>2</sup>    |
| Roll moment of inertia  | $I_{xx}$         | 0.319       | kg-m <sup>2</sup> |
| Pitch moment of inertia | $I_{yy}$         | 0.267       | kg-m <sup>2</sup> |
| Yaw moment of inertia   | $I_{zz}$         | 0.471       | kg-m <sup>2</sup> |
| Product of inertia      | $I_{xz}$         | 0.024       | kg-m <sup>2</sup> |
| Product of inertia      | $I_{xy}, I_{yz}$ | $\approx 0$ | kg-m <sup>2</sup> |

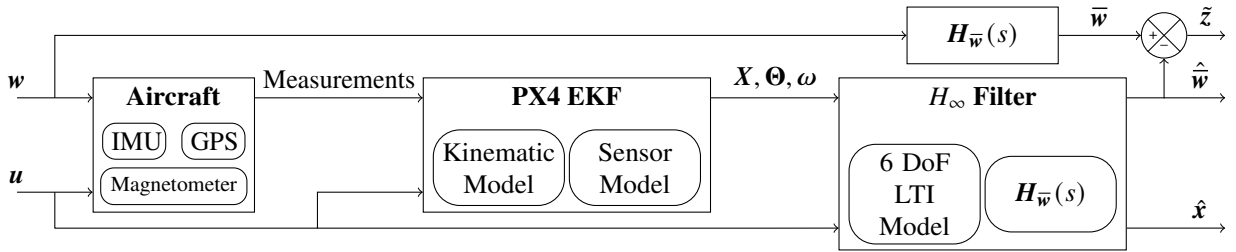


**Fig. 1** My Twin Dream (MTD) aircraft

The accelerometers, a gyroscope, magnetometers and GPS feed into an extended KALMAN filter as part of the PX4 firmware, from which we use position ( $X$ ), attitude ( $\Theta$ ), and angular velocity ( $\omega$ ). The PX4-provided  $X$ ,  $\Theta$ , and  $\omega$  estimates also include time-varying covariances [22]. For our purposes these measurement covariances,  $R$ , were averaged over the duration of the steady-maneuver, yielding

$$R = \text{diag}(0.0256, 0.0257, 0.1513, 0.0038, 0.0037, 0.0540, 0.0038, 0.0037, 0.0540) \quad (29)$$

The data sample period was  $T = 0.01$ s. The cascaded filter structure that is implemented on the MTD UAS is depicted in Fig. 2. The PX4 EKF estimates are provided to the  $H_\infty$  filter or the comparative KALMAN filter). The  $H_\infty$  filter incorporates the 6 degree of freedom (DoF) aircraft model and frequency filter model from Section III and produces estimates of the aircraft states and the estimated filtered wind. For validation, the  $H_\infty$  and KALMAN filter estimates are compared to the  $H_{\bar{w}}$ -filtered ADU wind measurements.



**Fig. 2** Cascade filter architecture.

The truth source for wind estimate validation was a small vaned air data unit (ADU) mounted out the nose of the aircraft, as pictured in Figure 1. This sensor was developed, manufactured, and calibrated by the Nonlinear Systems Laboratory at Virginia Tech [23]. It consists of two 3D printed vanes attached to micro magnetic rotary encoders and a 3D printed Kiel probe connected to a commercial off-the-shelf (COTS) pressure transducer. These sensors are

integrated into the PX4 firmware with a custom driver such that data is logged at a data rate of 200 Hz and timestamped along with other sensor data and EKF outputs. The air data unit provides *in situ* relative air velocity measurements. The sensor readings are corrected for vehicle angular velocity, and then using GPS velocity, the wind triangle (Eq. (9)) is used to obtain an instantaneous truth value for the apparent wind. This air data unit was validated in the Virginia Tech Aerospace and Ocean Engineering department's open jet wind tunnel both for steady-state readings and dynamic response. The airspeed-dependent cutoff frequencies of the vanes were found to be greater than 16 Hz, well above the fastest rigid body modes, which determine an upper limit on the dynamic response of model-based wind estimators.

## B. $H_\infty$ Filter Synthesis

The first step in synthesizing the  $H_\infty$  filter is to linearize Eq. (13) in preparation for creating the augmented system Eq. (21). To begin the linearization process, the trim states and controls of Eq. (13) are found via MATLAB using a publicly available local minimizing function, `fminsa`, which is based on [24]. The chosen trim condition, for the results presented here, is straight-and-level flight at a constant airspeed, heading, and altitude with zero wind. Alternatively, a different trim condition, such as a steady turn, could be chosen.

The trim states and controls, which determine equilibrium flight, are then used in a linearization routine to find the matrices of Eq. (17). The routine numerically approximates the JACOBIANS in Eq. (16) by adaptively reducing perturbation stepsize. The resultant matrices  $\mathbf{A}_A$  and  $\mathbf{B}_B$  are omitted for brevity. The matrix  $\mathbf{\Pi}$  is found to be  $\mathbb{I}^{3 \times 3}$ , where  $\mathbb{I}^{a \times a}$  is an  $a \times a$  identity matrix.  $\mathbf{B}_{\bar{w}}$  is dictated by the choice of filter from Eq. (18). For the present work, a first-order low-pass filter is implemented such that

$$\mathbf{H}_{\bar{w}}(s) = \begin{bmatrix} \frac{\omega_c}{s+\omega_c} & 0 & 0 \\ 0 & \frac{\omega_c}{s+\omega_c} & 0 \\ 0 & 0 & \frac{\omega_c}{s+\omega_c} \end{bmatrix} \quad (30)$$

where  $\omega_c$  is the desired cutoff frequency in rad/s. Therefore,  $\mathbf{G}$  is

$$\mathbf{G} = \begin{bmatrix} \mathbb{I}^{3 \times 3} & \mathbf{0}^{3 \times 9} \\ \mathbf{0}^{9 \times 3} & \mathbf{0}^{9 \times 9} \\ \mathbf{B}_{\bar{w}} & \mathbf{0}^{3 \times 9} \end{bmatrix} \quad (31)$$

where  $\mathbf{0}^{a \times b}$  is an  $a \times b$  matrix of zeros. The  $\mathbf{C}_A$  and  $\mathbf{D}$  matrices are

$$\mathbf{C}_A = \begin{bmatrix} \mathbb{I}^{3 \times 3} & \mathbf{0}^{3 \times 3} & \mathbf{0}^{3 \times 3} & \mathbf{0}^{3 \times 3} & \mathbf{0}^{3 \times 3} \\ \mathbf{0}^{3 \times 3} & \mathbb{I}^{3 \times 3} & \mathbf{0}^{3 \times 3} & \mathbf{0}^{3 \times 3} & \mathbf{0}^{3 \times 3} \\ \mathbf{0}^{3 \times 3} & \mathbf{0}^{3 \times 3} & \mathbf{0}^{3 \times 3} & \mathbb{I}^{3 \times 3} & \mathbf{0}^{3 \times 3} \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} \mathbf{0}^{9 \times 4} \end{bmatrix} \quad (32)$$

since, for the avionics system described in Section IV.A, we have direct access to measurements of inertial position, EULER angles, and body angular rates. To simplify the comparison between the  $H_\infty$  filter and the KALMAN filter in Section IV.C, we choose  $\mathbf{\Lambda} = \mathbf{R}_c$ . More generally,  $\mathbf{\Lambda}$  can be chosen by the user to describe the relative magnitude of measurement disturbances if that information is known. With our choice, the matrix  $\mathbf{F}$  is

$$\mathbf{F} = \begin{bmatrix} \mathbf{0}^{9 \times 3} & \mathbf{R}_c \end{bmatrix} \quad (33)$$

where  $\mathbf{R}_c = \mathbf{R} \cdot T$  is the continuous-time measurement noise spectral density [20] and  $\mathbf{R}$  and  $T$  are the discrete-time measurement covariance and sample time, respectively, provided by the avionics system described in Section IV.A. For this work it is assumed that the only states we wish to estimate are the mean wind states:

$$\mathbf{z} = \bar{\mathbf{w}} \quad (34)$$

Thus the desired state estimate combination  $\mathbf{L}$  from Eq. (21c) is

$$\mathbf{L} = \begin{bmatrix} \mathbf{0}^{3 \times 12} & \mathbf{C}_{\bar{w}} \end{bmatrix} \quad (35)$$

With all of the required matrices identified, Eq. (27) is solved using CVX in MATLAB with the SDPT3 solver [25–27]. For the example presented here the worst-case disturbance amplification is  $\gamma = 0.108$ . Recalling Lemma 1 this means

$$\|\mathbf{H}_{\tilde{z}\eta}(s)\|_\infty = \sup_{\eta \in \mathcal{L}_2} \frac{\|\tilde{z}\|_2^2}{\|\eta\|_2^2} < 0.108 \quad (36)$$

That is, the map from exogenous inputs to estimate error is contractive with a worst-case gain of 0.108.

### C. Comparative KALMAN Filter

To provide a comparison for the  $H_\infty$  filter, a steady-state KALMAN filter was developed according to [20, 28]. The steady-state KALMAN filter is found by solving the continuous algebraic RICCATI equation (ARE)

$$\mathbf{A}_A \mathbf{P} + \mathbf{P} \mathbf{A}_A^\top + \mathbf{\Pi}_A \mathbf{Q}_c \mathbf{\Pi}_A^\top - \mathbf{P} \mathbf{C}_A^\top \mathbf{R}_c^{-1} \mathbf{C}_A \mathbf{P} = \mathbf{0} \quad (37)$$

where  $\mathbf{P} = \mathbf{P}^\top > \mathbf{0}$  and  $\mathbf{Q}_c = \mathbf{Q}/T$  and where  $\mathbf{Q}$  is the discrete-time process noise covariance. Typically  $\mathbf{Q}$  is determined via trial and error or using engineering intuition. For the present problem, however, the only process noise we are considering is the wind disturbance. Therefore the choice of  $\mathbf{Q}$  is informed by a few assumptions which, we note, are unneeded in the design of the  $H_\infty$  filter. First, we assume that the vertical wind,  $w_D$ , at low altitudes (i.e. in the atmospheric boundary layer) is zero-mean. Then we assume that the wind turbulence is isotropic. Although this is not necessarily true at low altitude, it is our best guess for the design of the KALMAN filter. Thus we can find  $\mathbf{Q}$  from the variance of  $w_D$ . Keep in mind that this method of informing  $\mathbf{Q}$  requires a reliable source of measured wind. This information will not normally be available. For the comparison given here, though, the KALMAN filter is designed using all available information except for the direct wind measurements from the ADU that are reserved for validation.

The ARE of Eq. (37) is solved using the MATLAB function `icare` ( $\mathbf{A}_A^\top, \mathbf{C}_A^\top, \mathbf{\Pi}_A \mathbf{Q}_c \mathbf{\Pi}_A^\top, \mathbf{R}_c$ ). Finally, the KALMAN gain is found from

$$\mathbf{K}_K = \mathbf{P} \mathbf{C}_A^\top \mathbf{R}_c^{-1} \quad (38)$$

The KALMAN filter is implemented in the same manner as the  $H_\infty$  filter, that is using Eq. (22). Thus, the only difference between the KALMAN filter and the  $H_\infty$  filter presented in this work is the observer gain matrix  $\mathbf{K}_K$  versus  $\mathbf{K}_\infty$ . The gain matrices are obtained using distinctly different theoretical approaches, however, and as shown in the following section, the performance of these filters is significantly different.

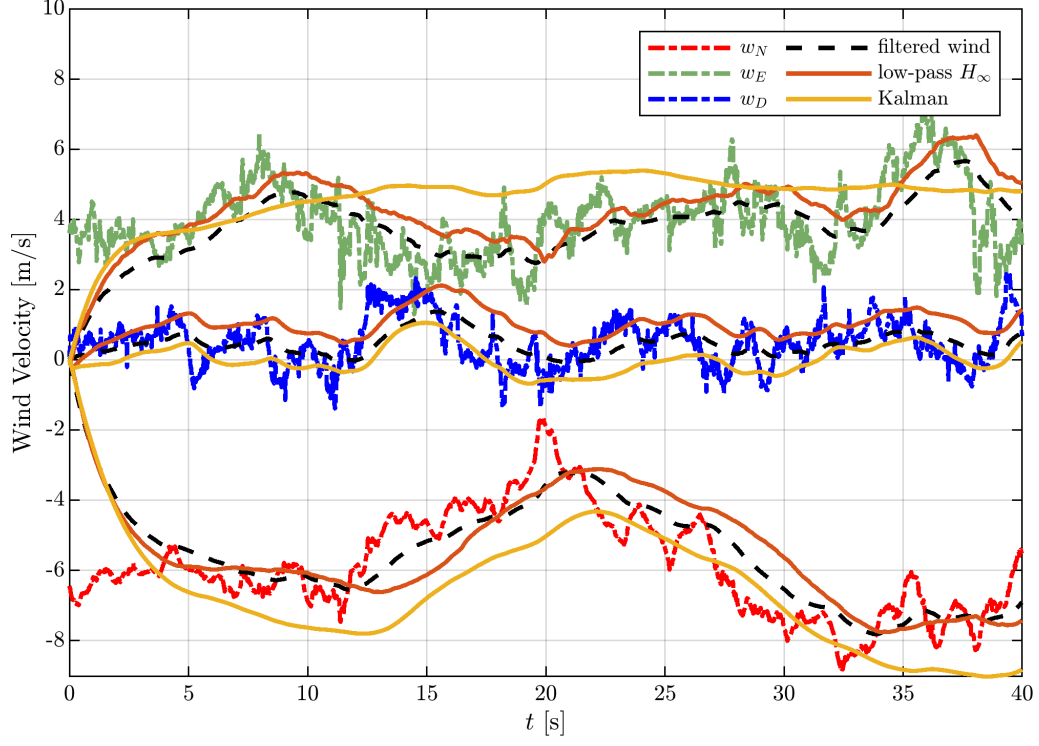
## V. Flight Test Results

A flight test campaign was conducted at Virginia Tech’s Kentland Experimental Aerial Systems (KEAS) airfield on September 28th, 2022 using the UAS described in Section IV.A. Data were gathered during four passes over the ground control station (referred to as Runs 1-4) at 76 m (250 ft) AGL when the wind conditions at ground level were gusty. Gusty conditions were chosen since the goal is to achieve accurate wind estimates under any condition in which the UAS is able to fly. Since we are utilizing an LTI model we restricted our maneuver to nominally steady flight, namely straight-and-level unaccelerated flight. To evaluate the performance of the filters three different cutoff frequencies  $\omega_c \in \{0.5, 2, 5\}$  rad/s were chosen to represent slow, medium, and fast wind fluctuations, respectively [29].

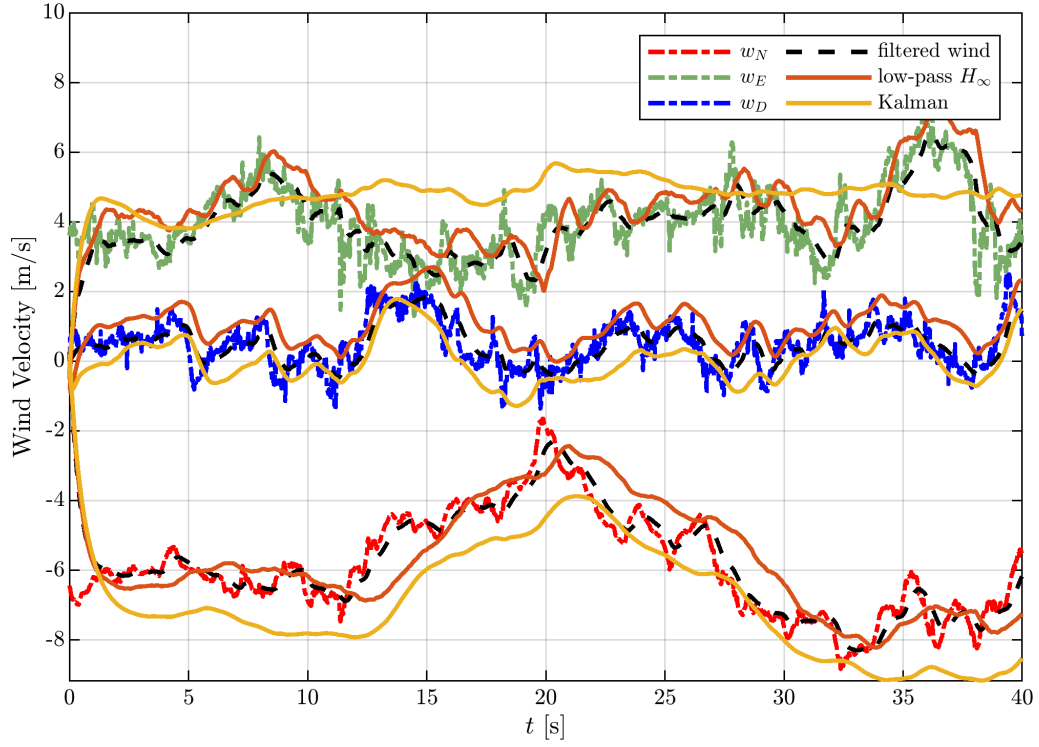
Figs. 3-5 show the results of the  $H_\infty$  and KALMAN filters, as applied to the system in Eq. (20), using Run 1 from flight test data. Run 1 was chosen because it depicts the worst performance of the  $H_\infty$  filter versus the KALMAN filter, since the KALMAN filter more accurately estimates  $w_D$ . For validation the filter estimates are compared to low-pass filtered ADU wind measurements.

In Fig. 3 the filter cutoff frequency is 0.5 rad/s, which corresponds to the relatively slow bulk flow of the wind. As can be seen the  $H_\infty$  filter more closely tracks the filtered wind shown in the dashed black line in all three directions compared to the KALMAN filter. Figs. 4 and 5 present the results using a cutoff frequency of 2 and 5 rad/s, respectively, and thus capture higher frequency fluctuations. In both cases the  $H_\infty$  filter is able to accurately track the actual wind fluctuations within the specified frequency band.

Lastly, Table 2 presents the root-mean-square (RMS) error between the filter estimates and the low-pass filtered direct measurements for all four runs. The  $H_\infty$  filter estimates are more accurate in all three directions and, in the case of the horizontal components of the wind ( $w_N$  and  $w_E$ ), have roughly half the error of the corresponding KALMAN filter estimate.



**Fig. 3** Results from Run 1 with  $\omega_c = 0.5$  rad/s



**Fig. 4** Results from Run 1 with  $\omega_c = 2$  rad/s

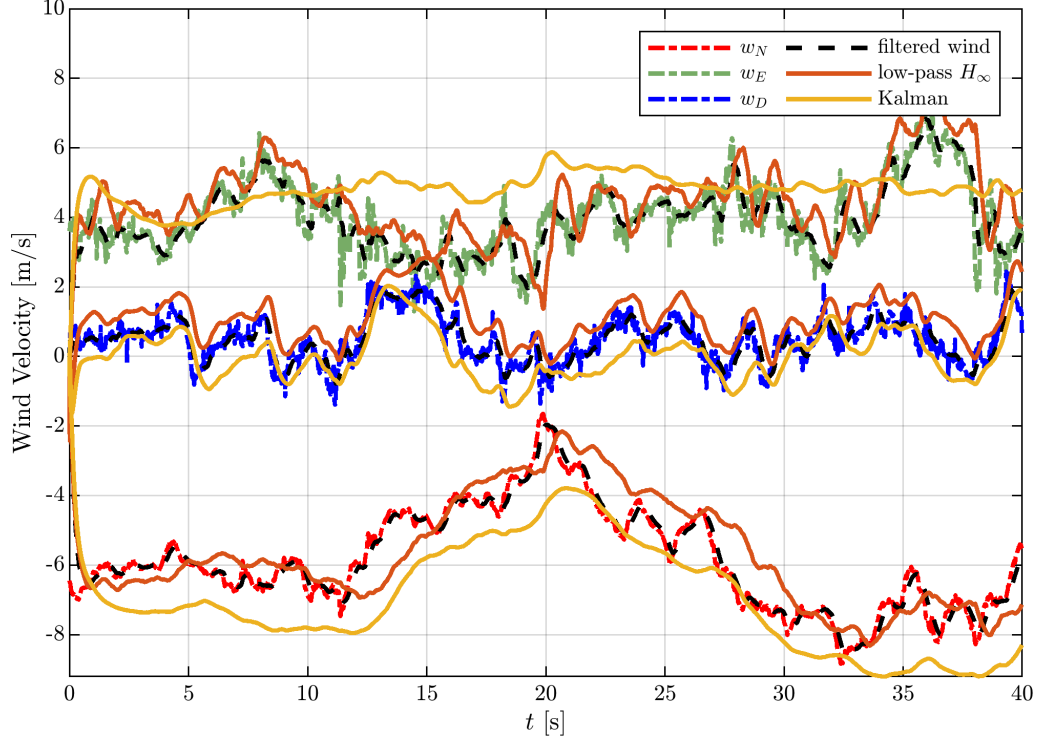


Fig. 5 Results from Run 1 with  $\omega_c = 5$  rad/s

Table 2 Root-mean-square error between filter estimates and direct low-pass filter of measured wind

| Filter     | Wind Direction | Cutoff Freq. ( $\omega_c$ ) [rad/s] | Flight Test Run (error in [m/s]) |       |       |       | Average Error [m/s] |
|------------|----------------|-------------------------------------|----------------------------------|-------|-------|-------|---------------------|
|            |                |                                     | 1                                | 2     | 3     | 4     |                     |
| $H_\infty$ | $w_N$          | 0.5                                 | 0.461                            | 0.710 | 0.364 | 0.545 | 0.520               |
|            |                | 2                                   | 0.639                            | 0.907 | 0.486 | 0.621 | 0.663               |
|            |                | 5                                   | 0.730                            | 0.989 | 0.671 | 0.733 | 0.781               |
|            | $w_E$          | 0.5                                 | 0.545                            | 0.665 | 1.395 | 0.532 | 0.784               |
|            |                | 2                                   | 0.631                            | 0.727 | 1.487 | 0.590 | 0.859               |
|            |                | 5                                   | 0.741                            | 0.782 | 1.525 | 0.617 | 0.916               |
|            | $w_D$          | 0.5                                 | 0.567                            | 0.555 | 0.556 | 0.447 | 0.531               |
|            |                | 2                                   | 0.622                            | 0.609 | 0.608 | 0.542 | 0.595               |
|            |                | 5                                   | 0.657                            | 0.646 | 0.724 | 0.655 | 0.671               |
| KALMAN     | $w_N$          | 0.5                                 | 1.154                            | 1.317 | 0.958 | 0.932 | 1.090               |
|            |                | 2                                   | 1.271                            | 1.429 | 1.151 | 1.125 | 1.244               |
|            |                | 5                                   | 1.315                            | 1.469 | 1.212 | 1.165 | 1.290               |
|            | $w_E$          | 0.5                                 | 1.109                            | 1.132 | 1.576 | 1.601 | 1.355               |
|            |                | 2                                   | 1.274                            | 1.348 | 1.693 | 1.692 | 1.502               |
|            |                | 5                                   | 1.329                            | 1.403 | 1.737 | 1.715 | 1.546               |
|            | $w_D$          | 0.5                                 | 0.471                            | 0.617 | 0.555 | 0.721 | 0.591               |
|            |                | 2                                   | 0.540                            | 0.724 | 0.688 | 0.806 | 0.690               |
|            |                | 5                                   | 0.583                            | 0.765 | 0.755 | 0.886 | 0.747               |

## VI. Conclusions and Future Work

This paper describes the concatenation of a low-pass filter and an  $H_\infty$  state estimator to produce a flight dynamic model-based estimate of the variations in wind velocity experienced by an aircraft. The cutoff frequency of the low-pass filter can be adjusted to determine what frequency band of wind fluctuations should be retained in the estimate. Unlike a KALMAN filter, the  $H_\infty$  filter does not require statistical assumptions about the process noise (i.e. the wind) and measurement noise and is thus more applicable to wind estimation where the wind fluctuations cannot be assumed to be white, GAUSSIAN noise.

The proposed  $H_\infty$  filter, via a bounded mapping from the disturbance inputs to the wind estimate error, provides a guarantee on the worst-case error, in contrast to the KALMAN filter. Additionally, the low-pass filter provides for flexibility in specifying the characteristics of the wind variations one wishes to estimate. The combined low-pass filter and  $H_\infty$  filter offers an effective means of obtaining accurate wind estimates within a desired frequency band while requiring minimal assumptions about the spectral character of the wind disturbances.

Future efforts will look to relax the assumptions on the wind field, namely to include rotational components of the wind or to remove the frozen turbulence assumption. It would also be worthwhile to consider more sophisticated filtering methods, such as the extended  $H_\infty$  filter [30, 31] or the unscented  $H_\infty$  filter [32], or to apply  $H_\infty$  smoothing techniques [33, 34].

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